Governing equations for 1D pipe networks: Continuity at nodes

- Mass balance: For each interior node i, the algebraic sum of inflows/outflows equals the specified demand d_i.

$$\sum_{j \in \mathcal{N}(i)} Q_{ij} = d_i$$

- Here Q_{ij} is positive if flow leaves node i toward node j. These nodal equations are the backbone of pipe-network analysis.

Head–loss along pipes - Energy relation: For a pipe between nodes i and j,

$$H_i - H_j = h_{ij}(Q_{ij})$$

- A common turbulent model is Darcy–Weisbach style with

$$h_{ij}(Q_{ij}) = R_{ij}\, Q_{ij}\ |Q_{ij}|$$

- where R_{ij} is the resistance computed from pipe length, diameter, friction factor, and gravity. These head-loss equations, combined with continuity, define looped network behavior.
- Boundary heads: Reservoirs or fixed-head nodes impose $H = H_{\text{given}}$ replacing continuity at those nodes with Dirichlet conditions.

The continuity at nodes and head-loss around loops are the standard laws used in looped network analysis; the overall setup is the classical pipe network analysis problem.

Linearization and pressure correction

Iterate from a current guess $(H^{(k)}, Q^{(k)})$ where the pressure-correction idea is to derive a linear relation between head corrections ΔH and flow corrections ΔQ using a first-order expansion of the head-loss.

- Linearize head-loss:

$$h_{ij}(Q_{ij}^{(k)} + \Delta Q_{ij}) \approx h_{ij}(Q_{ij}^{(k)}) + \left.\frac{\partial h_{ij}}{\partial Q}\right|_{Q_{ij}^{(k)}} \Delta Q_{ij}$$

- For

$$h_{ij} = R_{ij} Q_{ij}\, |Q_{ij}|$$

$$\left.\frac{\partial h_{ij}}{\partial Q}\right|_{Q_{ij}^{(k)}} = 2R_{ij}\left|Q_{ij}^{(k)}\right|$$

- Relate head corrections to flow corrections: - Using $H_i - H_j = h_{ij}$

$$\Delta H_i - \Delta H_j = \left.\frac{\partial h_{ij}}{\partial Q}\right|_{Q_{ij}^{(k)}} \Delta Q_{ij} \quad \Rightarrow \quad \Delta Q_{ij} = \beta_{ij}\,(\Delta H_i - \Delta H_j)$$

- with the pipe 'conductance'

$$\beta_{ij} = \frac{1}{\left.\frac{\partial h_{ij}}{\partial Q}\right|_{Q_{ij}^{(k)}}} = \frac{1}{2R_{ij}\left|Q_{ij}^{(k)}\right|}$$

This is the pressure-correction link: local slopes of the head–flow curve define a linear mapping from head corrections to flow corrections, much like pressure-correction schemes in CFD that linearize momentum–pressure coupling to enforce mass conservation.

Nodal correction equation derivation

Define the continuity residual at node i using the current flows:

$$b_i^{(k)} = d_i - \sum_{j\in\mathcal{N}(i)} Q_{ij}^{(k)}$$

We want to choose ΔH so that the corrected flows satisfy continuity:

$$\sum_{j\in\mathcal{N}(i)} (Q_{ij}^{(k)} + \Delta Q_{ij}) = d_i \quad \Rightarrow \quad \sum_{j\in\mathcal{N}(i)} \Delta Q_{ij} = b_i^{(k)}$$

Substitute the linear relation for each incident pipe:

$$\sum_{j\in\mathcal{N}(i)} \beta_{ij}\,(\Delta H_i - \Delta H_j) = b_i^{(k)}$$

Rearrange into a symmetric nodal system:

$$\left(\sum_{j\in\mathcal{N}(i)} \beta_{ij}\right)\Delta H_i - \sum_{j\in\mathcal{N}(i)} \beta_{ij}\,\Delta H_j = b_i^{(k)}$$

Collect over all interior nodes to form $[\mathbf{A}]\ \ \{\Delta\mathbf{H}\} = \{\mathbf{b}\}$ where $A_{ii} = \sum_{j\in\mathcal{N}(i)} \beta_{ij}$ and $A_{ij} = -\beta_{ij}$ for neighbors $j \in \mathcal{N}(i)$ - rows for fixed-head nodes are removed or enforced via Dirichlet conditions. Solve for $\Delta\mathbf{H}$, update heads $H^{(k+1)} = H^{(k)} + \Delta H$ then update flows using

$$Q_{ij}^{(k+1)} = Q_{ij}^{(k)} + \beta_{ij}\,(\Delta H_i - \Delta H_j)$$

Iterate until $\|\mathbf{b}^{(k)}\|$ is sufficiently small. This construction enforces nodal continuity while respecting the linearized energy relation on each pipe.

Practical algorithm

- Initialization: Choose initial nodal heads $H^{(0)}$ (such as near source heads) and compute initial flows by solving

$$H_i - H_j = R_{ij}Q_{ij}|Q_{ij}|$$

per pipe with consistent signs. - Standard practice starts from a guess and iteratively enforces continuity and head-loss laws in looped networks.

- At each iteration k:

- Compute residuals:

$$b_i^{(k)} = d_i - \sum_j Q_{ij}^{(k)}$$

- Build conductances:

$$\beta_{ij} = \frac{1}{2R_{ij}\left|Q_{ij}^{(k)}\right|}$$

- Assemble nodal matrix:

$$A_{ii} = \sum_j \beta_{ij}, A_{ij} = -\beta_{ij}$$

- Apply boundaries: Enforce fixed heads by Dirichlet treatment.
- Solve correction:

$$[\mathbf{A}]\{\Delta\mathbf{H}\} = \{\mathbf{b}\}$$

- Update:

$$H^{(k+1)} = H^{(k)} + \Delta H$$

$$Q_{ij}^{(k+1)} = Q_{ij}^{(k)} + \beta_{ij}(\Delta H_i - \Delta H_j)$$

- Convergence check: Stop when mass imbalance and head changes are below tolerance.

- Notes on robustness:

- Under-relaxation: Blend updates to stabilize when flows are near zero and derivatives blow up.

- Zero-flow protection: If $|Q_{ij}^{(k)}|$ is too small, use a laminar linear resistance

$$h_{ij} = R_{ij}^{(\ell)} Q_{ij}$$

with $\beta_{ij} = 1/R_{ij}^{(\ell)}$, or cap β_{ij}

- Friction updates: Refresh R_{ij} each iteration if friction depends on Reynolds number.

The iterative correction to satisfy continuity mirrors pressure-correction ideas in CFD, where a pressure equation is derived from mass conservation and linearized momentum; here, conductances arise from the local slope of the pipe head-loss law, while the network laws remain those used in classical looped network analysis.

Assumptions and variants

- Flow regime: The derivative $\partial h/\partial Q$ depends on the chosen head-loss model (Darcy–Weisbach or Hazen–Williams). The derivation is identical: only R_{ij} and its dependence on Q change.
- Compressibility: Assumes incompressible liquid and steady-state demands; for gas networks, include pressure-squared formulations and linearize accordingly.
- Elevation heads: If elevation differs, use total head $H = p/\gamma + z$ where elevation terms appear in the energy equation per pipe but do not alter the correction structure.