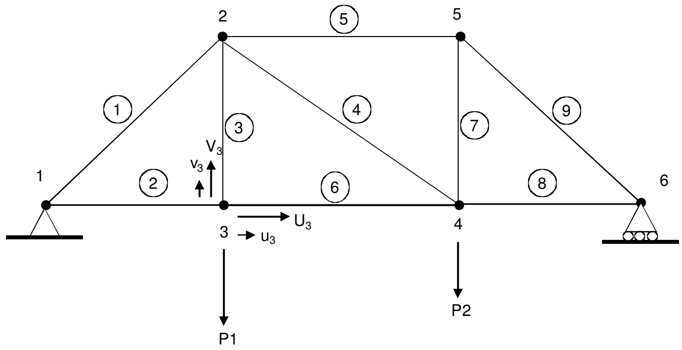
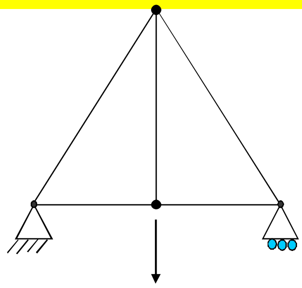
Typical Steps for FE Calculation of an Indeterminate Truss:





Forces Ui, Vi are related to the nodal displacement components as:

$$\begin{Bmatrix} U_1 \\ V_1 \\ . \\ . \\ . \\ U_6 \\ V_6 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ . \\ . \\ . \\ u_6 \\ v_6 \end{Bmatrix}$$

where [K] is the stiffness matrix for the entire truss (termed as Global Stiffness Matrix in Finite Elements).

$$\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \\ U_5 \\ V_5 \\ U_6 \\ V_6 \end{Bmatrix} = \begin{pmatrix}
K_{U1\,u1} & K_{U1\,v1} & K_{U1\,u2} & K_{U1\,v2} & K_{U1\,u3} & K_{U1\,v3} & K_{U1\,u4} & K_{U1\,v4} & K_{U1\,u5} & K_{U1\,v5} & K_{U1\,u6} & K_{U1\,v6} \\
K_{V1\,u1} & K_{V1\,v1} & K_{V1\,u2} & K_{V1\,v2} & K_{V1\,u3} & K_{V1\,v3} & K_{V1\,u4} & K_{V1\,v4} & K_{V1\,u5} & K_{V1\,v5} & K_{V1\,u6} & K_{V1\,v6} \\
K_{U2\,u1} & K_{U2\,v1} & K_{U2\,u2} & K_{U2\,v2} & K_{U2\,u3} & K_{U2\,v3} & K_{U2\,u4} & K_{U2\,v4} & K_{U2\,u5} & K_{U2\,v5} & K_{U2\,u6} & K_{U2\,v6} \\
K_{V2\,u1} & K_{V2\,v1} & K_{V2\,u2} & K_{V2\,v2} & K_{V2\,u3} & K_{V2\,v3} & K_{V2\,u4} & K_{V2\,v4} & K_{V2\,u5} & K_{V2\,v5} & K_{V2\,u6} & K_{V2\,v6} \\
K_{U3\,u1} & K_{U3\,v1} & K_{U3\,u2} & K_{U3\,v2} & K_{U3\,u3} & K_{U3\,v3} & K_{U3\,u4} & K_{U3\,v4} & K_{U3\,u5} & K_{U3\,v5} & K_{U3\,u6} & K_{U3\,v6} \\
K_{V3\,u1} & K_{V3\,v1} & K_{V3\,u2} & K_{V3\,v2} & K_{V3\,u3} & K_{V3\,v3} & K_{V3\,u4} & K_{V3\,v4} & K_{V3\,u5} & K_{V3\,v5} & K_{V3\,u6} & K_{V3\,v6} \\
K_{U4\,u1} & K_{U4\,v1} & K_{U4\,u2} & K_{U4\,v2} & K_{U4\,u3} & K_{U4\,v3} & K_{U4\,u4} & K_{U4\,v4} & K_{U4\,u5} & K_{U4\,v5} & K_{U4\,u6} & K_{U4\,v6} \\
K_{V4\,u1} & K_{V4\,v1} & K_{V4\,u2} & K_{V4\,v2} & K_{V4\,u3} & K_{V4\,v3} & K_{V4\,u4} & K_{V4\,v4} & K_{V4\,u5} & K_{V4\,v5} & K_{V4\,u6} & K_{V4\,v6} \\
K_{U5\,u1} & K_{U5\,v1} & K_{U5\,u2} & K_{U5\,v2} & K_{U5\,u3} & K_{U5\,v3} & K_{U5\,u4} & K_{U5\,v4} & K_{U5\,u5} & K_{U5\,v5} & K_{U5\,u6} & K_{U5\,v6} \\
K_{V5\,u1} & K_{V5\,v1} & K_{V5\,u2} & K_{V5\,v2} & K_{V5\,u3} & K_{V5\,v3} & K_{V5\,u4} & K_{V5\,v4} & K_{V5\,u5} & K_{V5\,v5} & K_{V5\,u6} & K_{V5\,v6} \\
K_{U6\,u1} & K_{U6\,v1} & K_{U6\,u2} & K_{U6\,v2} & K_{U6\,u3} & K_{U6\,v3} & K_{U6\,u4} & K_{U6\,v4} & K_{U6\,u5} & K_{U6\,v5} & K_{U6\,u6} & K_{U6\,v6} \\
K_{V6\,u1} & K_{V6\,v1} & K_{V6\,u2} & K_{V6\,v2} & K_{V6\,u3} & K_{V6\,v3} & K_{V6\,u4} & K_{V6\,v4} & K_{V6\,u5} & K_{V6\,v5} & K_{V6\,u6} & K_{V6\,v6}
\end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}$$

Here $K_{Ui\,ui}$ is horizontal force component from pin i needed for a unit horizontal deflection of pin i with all other nodal displacement components kept at zero. Similarly, $K_{Vp\,vq}$ is the vertical force component from pin P needed for a unit vertical deflection of pin q with all other nodal displace- -ment component held at zero.

Let us define $\{q_i\}$ as the force vector from node i onto the members $\{q_i\} = \begin{Bmatrix} U_i \\ V_i \end{Bmatrix}$

Define {ai} as the nodal displacement vecotr at node i. $\{a_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$

$$\begin{Bmatrix} \{q_1\} \\ . \\ . \\ \{q_6\} \end{Bmatrix} = \overset{[K]^e}{\begin{bmatrix} [K_{11}] & . & . & . & [K_{16}] \\ . & . & . & . & . \\ . & . & . & . & . \\ [K_{61}] & . & . & . & [K_{66}] \end{bmatrix}} \begin{Bmatrix} \{a_1\} \\ . \\ . \\ \{a_6\} \end{Bmatrix}$$

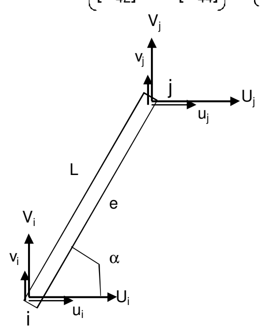
$$[K_{11}] = \begin{pmatrix} K_{U1\,u1} & K_{U1\,v1} \\ K_{V1\,u1} & K_{V1\,v1} \end{pmatrix}$$

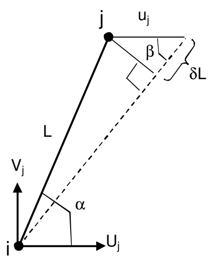
$$\Downarrow$$

$$[K_{ij}] = \begin{pmatrix} K_{Ui\,uj} & K_{Ui\,vj} \\ K_{Vi\,uj} & K_{Vi\,vj} \end{pmatrix}$$

For an element of the truss, say element 4, the relation between forces and displacements on nodes 2 & 4 are:

$$\begin{Bmatrix} \{q_2\} \\ \{q_4\} \end{Bmatrix} = \begin{pmatrix} [K_{22}] & [K_{24}] \\ [K_{42}] & [K_{44}] \end{pmatrix} \begin{Bmatrix} \{a_2\} \\ \{a_4\} \end{Bmatrix}$$





Let us induce a small displacement u_j, keeping other nodal displacement components equal to zero. The elongation δL of the member is then

$\delta L = u_j \cos\beta$ For small displacement u_j, we can take angle $\alpha \approx \beta$. Hence, $\delta L \approx u_j \cos\alpha$.

With modulus of elasticity E, the force P in the member is easily determined as $P = \delta L/L.EA = u_j \cos\alpha\, EA / L$ (tension) where A is the cross-sectional area of the member. We can now express $K_{Ui\,uj}$ as

$$K_{Uiuj} = \frac{-P\cos\alpha}{u_j} = \frac{-\cos^2\alpha}{L/EA}$$

U_i is the force on the member from pin i to be associated with unit nodal displacement component u_j at pin j. Since member ij will be in tension for u_j, U_i must be in the negative x direction so that K_{Uiuj} is negative.

Similarly,

$$K_{Viuj} = \frac{-P\sin\alpha}{u_j} = \frac{-\sin\alpha\cos\alpha}{L/EA}$$

Now if we consider the forces U_i and V_j from u_j at the other nodal point j, we get the same results as for U_i and V_i except for sign, as can readily be deduced from equilibrium. We can accordinagly say for u_j that

$$K_{Ui\,uj} = \frac{-\cos^2\alpha}{L/EA} \qquad K_{Vi\,uj} = \frac{-\sin 2\alpha}{2L/EA} \qquad K_{Uj\,uj} = \frac{+\cos^2\alpha}{L/EA} \qquad K_{Vj\,uj} = \frac{+\sin\alpha}{2L/EA}$$

If we now consider u_i (i.e. u at the other nodal point), we get compression of the member. It is easily seen that we get the above results with changed sign and with u_j replaced by u_i. Thus:

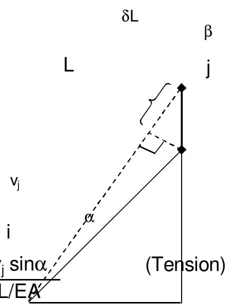
$$K_{Ui\,ui} = \frac{+\cos^2\alpha}{L/EA} \qquad K_{Vi\,ui} = \frac{+\sin 2\alpha}{2L/EA} \qquad K_{Uj\,ui} = \frac{-\cos^2\alpha}{L/EA} \qquad K_{Vj\,ui} = \frac{-\sin\alpha}{2L/EA}$$

Now consider v_j:

$\delta L = v_j \cos\beta$

For small displacement u_j, we can take angle $\alpha \approx \beta$. Hence,

$\delta L \approx v_j \sin\alpha$.



For a force P in member we have,

$$P = \frac{v_j \sin\alpha}{L/EA} \quad \text{(Tension)}$$

We can noe write:

$$K_{Ui\,vj} = \frac{-P\cos\alpha}{v_j} = \frac{-\sin 2\alpha}{2L/EA}$$

Similarly,

$$K_{Vi\,vj} = \frac{-P\sin\alpha}{v_j} = \frac{-\sin^2\alpha}{L/EA}$$

For Uj and Vi respectively, we get the same results as above except for sign (as a result of equilibrium), so that for vj we have two additional equations:

$$K_{Uj\,vj} = \frac{+\sin 2\alpha}{2L/EA} \qquad K_{Vj\,vj} = \frac{+\sin^2\alpha}{L/EA}$$

And for v_i we get compression in the member. It is easily seen again that we get the same result as above but with opposite signs and with v_i repacing v_j. Thus

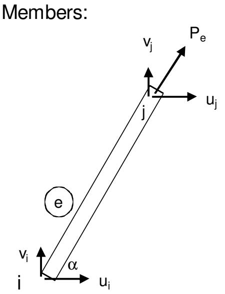
$$K_{Ui\,vi} = \frac{+\sin 2\alpha}{2L/EA} \qquad K_{Vi\,vi} = \frac{+\sin^2\alpha}{L/EA} \qquad K_{Uj\,vi} = \frac{-\sin 2\alpha}{2L/EA} \qquad K_{Vj\,vi} = \frac{-\sin^2\alpha}{L/EA}$$

The stiffness matrix for any element ij, with the <u>angle α measured counterclockwise</u> from the x axis to ij, can be given as follows:

2D Truss:

$$[K]^e = EA/L \begin{pmatrix} \cos^2\alpha & 1/2.\sin 2\alpha & -\cos^2\alpha & -1/2.\sin 2\alpha \\ 1/2.\sin 2\alpha & \sin^2\alpha & -1/2.\sin 2\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -1/2.\sin 2\alpha & \cos^2\alpha & 1/2.\sin 2\alpha \\ -1/2.\sin 2\alpha & -\sin^2\alpha & 1/2.\sin 2\alpha & \sin^2\alpha \end{pmatrix}$$

Forces in Members:



Force in member 'e':

$$Pe = \frac{AE}{L} [G]_e \begin{Bmatrix} \{a_i\} \\ \{a_j\} \end{Bmatrix}$$

$[G]_e = [-\cos\alpha, -\sin\alpha, \cos\alpha, \sin\alpha]$

3D Truss:

$$[K]^e = EA/L \begin{pmatrix} \cos^2\alpha_x & \cos\alpha_x \cos\alpha_y & \cos\alpha_x \cos\alpha_z & -\cos^2\alpha_x & -\cos\alpha_x \cos\alpha_y & -\cos\alpha_x \cos\alpha_z \\ \cos\alpha_x \cos\alpha_y & \cos^2\alpha_y & \cos\alpha_y \cos\alpha_z & -\cos\alpha_x \cos\alpha_y & -\cos^2\alpha_y & -\cos\alpha_y \cos\alpha_z \\ \cos\alpha_x \cos\alpha_z & \cos\alpha_y \cos\alpha_z & \cos^2\alpha_z & -\cos\alpha_x \cos\alpha_z & -\cos\alpha_y \cos\alpha_z & -\cos^2\alpha_z \\ -\cos^2\alpha_x & -\cos\alpha_x \cos\alpha_y & -\cos\alpha_x \cos\alpha_z & \cos^2\alpha_x & \cos\alpha_x \cos\alpha_y & \cos\alpha_x \cos\alpha_z \\ -\cos\alpha_x \cos\alpha_y & -\cos^2\alpha_y & -\cos\alpha_y \cos\alpha_z & \cos\alpha_x \cos\alpha_y & \cos^2\alpha_y & \cos\alpha_y \cos\alpha_z \\ -\cos\alpha_x \cos\alpha_z & -\cos\alpha_y \cos\alpha_z & -\cos^2\alpha_z & \cos\alpha_x \cos\alpha_z & \cos\alpha_y \cos\alpha_z & \cos^2\alpha_z \end{pmatrix}$$

$\cos\alpha_x = (Xi - Xj)\ /L$ L = Length of the truss

$\cos\alpha_y = (Yi - Yj)\ /L$

$\cos\alpha_z = (Zi - Zj)\ /L$